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Variations on theme of Nested Radicals (Inequalities, Recurrences, Boundness and Limits)

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ABSTRACT. By anology with continued fraction we will consider for given sequences $(p_n), (a_n), (b_n)$ finite and infinite "additive" and "multiplicative" Radical Constructions:

(SF)

$$\sqrt[p_1]{a_1 + b_1 \sqrt[p_2]{a_2 + b_2 \sqrt[p_3]{a_3 + ... + b_n \sqrt[p_{n+1}]{a_{n+1}}}}},$$

(SI)

$$\sqrt[p_1]{a_1 + b_1 \sqrt[p_2]{a_2 + b_2 \sqrt[p_3]{a_3 + ... + b_n \sqrt[p_{n+1}]{a_{n+1} + ...}}}}$$

(PF)

$$\sqrt[p_1]{a_1 \sqrt[p_2]{a_2 \sqrt[p_3]{a_3 \sqrt[p_{n+1}]{a_{n+1}}}}},$$

(PI)

$$\sqrt[p_1]{a_1 \sqrt[p_2]{a_2 \sqrt[p_3]{a_3 \sqrt[p_{n+1}]{a_{n+1} + ...}}}}$$

which named, respectively, finite and infinite nested (continued) radicals (additive and multiplicative). As usual, the basis for the variations will be concrete problems.

Part 1. Inequalities and boundedness.

Problem1.

a) Prove that $r_n := \sqrt{2\sqrt{3\sqrt{4\sqrt{....\sqrt{n+1}}}}} < 3, n \in \mathbb{N}$;

b) Prove that $r_n := \sqrt{2\sqrt[3]{3\sqrt[4]{4\sqrt[5]{....\sqrt[n]{n}}}}} < 3, n \in \mathbb{N}$. ($r_1 = \sqrt[1]{1} = 1$).

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Solution. a)
Solution 1. Since

$$r_n = 2^{\frac{1}{2}} 3^{\frac{1}{2^2}} 4^{\frac{1}{2^3}} \ldots. (n+1)^{\frac{1}{2^n}} \iff r_n^{2^n} = 2^{2^{n-1}} \cdot 3^{2^{n-2}} \cdot 4^{2^{n-3}} \ldots. (n+1)^{2^0}$$

then, applying AM-GM Inequality we obtain

$$r_n^{2^n} \le \left(\frac{2 \cdot 2^{n-1} + 3 \cdot 2^{n-2} + \ldots + n \cdot 2^1 + (n+1) \cdot 2^0}{2^{n-1} + 2^{n-2} + \ldots + 2 + 1} \right)^{2^{n-1}+2^{n-2}+\ldots+2+1}.$$

Since $2^{n-1} + 2^{n-2} + \ldots + 2 + 1 = 2^n - 1$,

$$2 \cdot 2^{n-1} + 3 \cdot 2^{n-2} + \ldots + n \cdot 2^1 + (n+1) \cdot 2^0 = 3 \cdot 2^n - n - 2$$

then

$$r_n^{2^n} \le \left(\frac{3 \cdot 2^n - n - 2}{2^n - 1} \right)^{2^n - 1} = \left(3 - \frac{n-1}{2^n - 1} \right)^{2^n - 1} \implies r_n < 3^{\frac{2^n - 1}{2^n}} < 3.$$

Solution 2. Since

$$\ln r_n = \frac{\ln 2}{2} + \frac{\ln 3}{2^2} + \ldots + \frac{\ln(n+1)}{2^n}$$

and for any natural k holds inequality

$$\ln(k+1) < 2\ln(k+2) - \ln(k+3) \iff$$

$$(k+1)(k+3) < (k+2)^2 \iff 0 < 1$$

then

$$\ln r_n = \sum_{k=1}^{n} \frac{\ln(k+1)}{2^k} < \sum_{k=1}^{n} \frac{2\ln(k+2) - \ln(k+3)}{2^k} =$$

$$= \sum_{k=1}^{n} \left(\frac{\ln(k+2)}{2^{k-1}} - \frac{\ln(k+3)}{2^k} \right) =$$

$$= \frac{\ln(1+2)}{2^{1-1}} - \frac{\ln(n+3)}{2^n} = \ln 3 - \frac{\ln(n+3)}{2^n} < \ln 3 \implies r_n < 3.$$

b)
Solution 1. Applying Weighted AM-GM Inequality to the numbers $2, 3, \ldots, n$ with weights

$$w_1 = \frac{1}{2!}, w_2 = \frac{1}{3!}, \ldots, w_{n-1} = \frac{1}{n!}$$

we obtain

$$r_n = 2^{\frac{1}{2!}} \cdot 3^{\frac{1}{3!}} \cdot \ldots \cdot n^{\frac{1}{n!}} < \left(\frac{2 \cdot \frac{1}{2!} + 3 \cdot \frac{1}{3!} + \ldots + n \cdot \frac{1}{n!}}{\frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}} \right)^{\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}} =$$

$$= \left(\frac{\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}}{\frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}} \right)^{\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}} <$$

$$< \left(1 + \frac{1}{\frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}} \right)^{\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}} < e.$$

Solution 2. Since $\ln n < n - 1, n \geq 2$ then

$$\ln r_n = \frac{\ln 2}{2!} + \frac{\ln 3}{3!} + \ldots + \frac{\ln n}{n!} < \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n-1}{n!} =$$

$$= \left(\frac{1}{1!} - \frac{1}{2!} \right) + \left(\frac{1}{2!} - \frac{1}{3!} \right) + \ldots + \left(\frac{1}{(n-1)!} - \frac{1}{n!} \right) =$$

$$= 1 - \frac{1}{n!} < 1 \implies r_n < e < 3.$$

Remark 1. *Better upper bound for r_n.*
Using more precise inequality $\ln n < n - 1, n \geq 2$ we obtain

$$\ln r_n = \frac{\ln 2}{2!} + \frac{\ln 3}{3!} + \ldots + \frac{\ln n}{n!} < \frac{\ln 2}{2} + \frac{\ln 4}{6} + \frac{\ln 4}{24} + \left(\frac{1}{4!} - \frac{1}{5!} \right) + \ldots + \left(\frac{1}{(n-1)!} - \frac{1}{n!} \right) =$$

$$= \frac{\ln 2}{2} + \frac{\ln 2}{3} + \frac{\ln 2}{12} + \frac{1}{4!} - \frac{1}{n!} < \frac{11 \ln 2}{12} + \frac{1}{24}.$$

Since

$$\frac{11 \ln 2}{12} + \frac{1}{24} < \ln 2 \iff \frac{1}{2} < \ln 2 \iff 1 < \ln 4$$

then $r_n < 2$.

The same upper bound for r_n gives

Solution 3. Since $n! \geq 2 \cdot 3^{n-2}$, $n \geq 2$ and $\max_{n \in \mathbb{N}} n^{\frac{1}{n}} = 3^{\frac{1}{3}}$ then for $k \geq 3$ holds

$$k^{\frac{1}{k!}} = \left(k^{\frac{1}{k}} \right)^{\frac{1}{(k-1)!}} \leq 3^{\frac{1}{3(k-1)!}} \leq 3^{\frac{1}{2 \cdot 3^{k-2}}}$$

and, therefore,

$$r_n = 2^{\frac{1}{2!}} \cdot 3^{\frac{1}{3!}} \cdot \ldots \cdot n^{\frac{1}{n!}} \leq 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2\cdot 3}} \cdot \ldots \cdot 3^{\frac{1}{2\cdot 3^{n-2}}} <$$

$$\sqrt{2 \cdot 3^{\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^{n-2}}+\ldots}} = \sqrt{2 \cdot 3^{\frac{1}{2}}} = \sqrt[4]{12} < 2.$$

Remark 2. *As generalization of considered above Problem 1 we will find upper bounds for*

$$r_n(k) := \sqrt[k]{k \sqrt[k+1]{(k+1) \ldots \sqrt[n]{n}}}$$

and $r(k) := \sqrt[k]{k \sqrt[k+1]{(k+1) \ldots \sqrt[n]{n \ldots}}}$

Lemma 1. For any natural numbers $n \geq 3$ and p holds following inequalities:

(I)

$$n^{\frac{1}{n^p}} > (n+1)^{\frac{1}{(n+1)^p}};$$

(II)

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)\ldots(n+p)}}.$$

Proof. (Using Math Induction by $p \in \mathbb{N}$) Inequality (I)

1. For $p = 1$ we already have $n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$.
2. For any $p \in \mathbb{N}$ assuming $n^{\frac{1}{n^p}} > (n+1)^{\frac{1}{(n+1)^p}}$ we obtain

$$n^{\frac{1}{n^{p+1}}} = \left(n^{\frac{1}{n^p}}\right)^{\frac{1}{n}} > \left((n+1)^{\frac{1}{(n+1)^p}}\right)^{\frac{1}{n}} > \left((n+1)^{\frac{1}{(n+1)^p}}\right)^{\frac{1}{n+1}} = (n+1)^{\frac{1}{(n+1)^{p+1}}}.$$

Inequality (II).

1. For $p = 1$ we already have $n^{\frac{1}{n}} > (n+1)^{\frac{1}{n+1}}$
2. For any $p \in \mathbb{N}$ using inequality (I) and assuming that inequality

$$n^{\frac{1}{n^p}} > (n+p)^{\frac{1}{(n+1)(n+2)\ldots(n+p)}}$$

holds for any $n \geq 3$ and we obtain

$$n^{\frac{1}{n^{p+1}}} = \left(n^{\frac{1}{n^p}}\right)^{\frac{1}{n}} > \left((n+1)^{\frac{1}{(n+1)^p}}\right)^{\frac{1}{n}} > \left(((n+1)+p)^{\frac{1}{((n+1)+1)\ldots(n+1+p)}}\right)^{\frac{1}{n}} >$$

$$> \left((n+1+p)^{\frac{1}{(n+2)\dots(n+1+p)}} \right)^{\frac{1}{n+1}} = (n+(p+1))^{\frac{1}{(n+1)(n+2)\dots(n+p+1)}}$$

Applying inequality (II) for $(n,p) = (k,p)$, where $p = 1,2,...,n-k$ to

$$r_n(k) := \sqrt[k]{k \sqrt[k+1]{(k+1) \dots \sqrt[n]{n}}}, 3 \le k \le n$$

we obtain

$$r_n(k) = k^{\frac{1}{k}} \cdot (k+1)^{\frac{1}{k(k+1)}} \cdot \dots \cdot n^{\frac{1}{k(k+1)\dots n}} =$$

$$= \quad k^{\frac{1}{k}} \cdot \left((k+1)^{\frac{1}{k+1}} \cdot (k+2)^{\frac{1}{(k+1)(k+2)}} \cdot \dots \cdot n^{\frac{1}{(k+1)\dots n}} \right)^{\frac{1}{k}} <$$

$$< \quad k^{\frac{1}{k}} \cdot \left(k^{\frac{1}{k}} \cdot k^{\frac{1}{k^2}} \cdot \dots \cdot k^{\frac{1}{k^{n-k}}} \right)^{\frac{1}{k}} = k^{\frac{1}{k} + \frac{1}{k^2} + \dots + \frac{1}{k^{n-k+1}}} < k^{\frac{1}{k-1}}.$$

So, $r_n(k) < k^{\frac{1}{k-1}}$ and since $r_n(k) \uparrow (n)$ then we have
$$r(k) := \lim_{n\to\infty} r_n(k) \le k^{\frac{1}{k-1}}.$$

Problem 2.

a) For any real $a > 0$ determine upper bound for
$a_n = \sqrt{a + \sqrt{a + \sqrt{a + \dots + \sqrt{a}}}}$(n-roots), $n \in \mathbb{N}$;

b) Let $a_n := \dfrac{\sqrt{n + \sqrt{n-1 + \sqrt{n-2 + \dots + \sqrt{1}}}}}{\sqrt{n}}, n \in \mathbb{N}.$

Prove that sequence $(a_n)_{\mathbb{N}}$ is bounded.

Solution.

a) Sequence $(a_n)_{\mathbb{N}}$ can be defined recursively as follows
$a_{n+1} = \sqrt{a + a_n}, n \in \mathbb{N}$ and $a_1 = \sqrt{a}$.
In supposition that positive number M is upper bound for $(a_n)_{\mathbb{N}}$ and since then

$$a_{n+1} = \sqrt{a + a_n} \le \sqrt{a + M}$$

we claim

$$\sqrt{a+M} \le M \iff a + M \le M^2 \iff M^2 - M - a \ge 0 \iff$$

$$M \ge \frac{1 + \sqrt{4a+1}}{2}.$$

Let

$$M := \frac{1+\sqrt{4a+1}}{2}.$$

Since

$$a_1 < M \iff \sqrt{a} < \frac{1+\sqrt{4a+1}}{2} \iff$$

$\sqrt{4a} \leq \sqrt{4a+1}+1$ obviously holds and for any $n \in \mathbb{N}$, assumption $a_n \leq M$ implies $a_{n+1} = \sqrt{a+a_n} \leq \sqrt{a+M} \leq M$, then by Math Induction $a_n \leq M$ for any natural n.

Remark. *If* $a = 2$ *then* $a_{n+1} = \sqrt{2+a_n}, n \in \mathbb{N}$ *where* $a_1 = \sqrt{2} = 2\cos\frac{\pi}{4}$ *and, therefore,*

$$a_2 = \sqrt{2+2\cos\frac{\pi}{2^2}} = 2\cos\frac{\pi}{2^3}.$$

For any $n \in \mathbb{N}$ assuming $a_n = 2\cos\frac{\pi}{2^{n+1}}$ we obtain

$$a_{n+1} = \sqrt{2+a_n} = \sqrt{2+2\cos\frac{\pi}{2^{n+1}}} = 2\cos\frac{\pi}{2^{n+2}}.$$

Thus, by Math Induction we have

$$a_n = 2\cos\frac{\pi}{2^{n+1}} < 2$$

for any $n \in \mathbb{N}$.

Formula $M = \dfrac{1+\sqrt{4a+1}}{2}$ for $a = 2$ gives us $M = 2$ as well.

b) Since

$$\sqrt{n+\sqrt{n-1+\sqrt{n-2+...+\sqrt{1}}}} > \sqrt{n}$$

then $a_n > 1$.

Note that for any $n \in \mathbb{N}$ holds inequality $a_{n+1} < \sqrt{1+a_n}$.

Indeed,

$$a_{n+1} = \frac{\sqrt{n+1+\sqrt{n+...+\sqrt{1}}}}{\sqrt{n+1}} \sqrt{1+\frac{1}{n+1}\sqrt{n+\sqrt{n-1+...+\sqrt{1}}}} <$$

$$< \sqrt{1+\frac{1}{\sqrt{n}}\sqrt{n-1+\sqrt{n-2+...+\sqrt{1}}}} = \sqrt{1+a_n}$$

For any $n \in \mathbb{N} \setminus \{1\}$ repeatedly applying this inequality we obtain

$$a_n < \sqrt{1+a_{n-1}} < \sqrt{1+\sqrt{1+a_{n-2}}} < ... < \sqrt{1+\sqrt{1+\sqrt{1+...+\sqrt{a_1}}}} =$$

$$= \sqrt{1+\sqrt{1+\sqrt{1+...+\sqrt{1}}}}$$

(n-roots) and,since

$$\sqrt{1+\sqrt{1+\sqrt{1+...+\sqrt{1}}}} \leq \frac{1+\sqrt{4\cdot 1+1}}{2} = \frac{1+\sqrt{5}}{2}$$

then $a_n < \dfrac{1+\sqrt{5}}{2}$
for any $n \in \mathbb{N}$.

Remark. Since

$$\sqrt{n+\sqrt{n-1+\sqrt{n-2+...+\sqrt{1}}}} < \sqrt{n}+1$$

for any $n \in \mathbb{N}$ (can be proved by Math Induction) then $a_n < \dfrac{\sqrt{n}+1}{\sqrt{n}} < 2$, for any $n \in \mathbb{N}$ and, therefore, $(a_n)_{\mathbb{N}}$ is bounded from above.

Problem 3. For any natural $n \geq 2$ prove inequality

$$\sqrt{2+\sqrt[3]{3+\sqrt[4]{4+\sqrt[5]{5+...+\sqrt[n]{n}}}}} < 2.$$

Solution. For any natural $n \geq 2$ let $r_0(n) = \sqrt[n]{n}$ and

$$r_k(n) = \sqrt[n-k]{n-k+r_{k-1}(n)},$$

where $0 \leq k \in \{1,2,...,n-1\}$. Then

$$r_1(n) = \sqrt[n-1]{n-1+r_n(0)} = \sqrt[n-1]{n-1+\sqrt[n]{n}}\ , r_2(n) = \sqrt[n-2]{n-2+r_1(n)} =$$

$$= \sqrt[n-2]{n-2+\sqrt[n-1]{n-1+\sqrt[n]{n}}}, .., r_k(n) =$$

$$= \sqrt[n-k]{n-k+\sqrt[n-(k+1)]{(n-(k+1))+...+\sqrt[n-1]{n-1+\sqrt[n]{n}}}},$$

$$\sqrt{2+\sqrt[3]{3+\sqrt[4]{4+\sqrt[5]{5+\ldots+\sqrt[n]{n}}}}}=r_{n-2}(n)$$

and we have to prove that $r_{n-2}(n)<2$.
For further we need the following

Lemma 2. For any $n\geq 3$ and real $h>0$ holds inequality

$$\sqrt[n]{n+h}\geq \sqrt[n+1]{n+1+h}.$$

Proof. We have

$$\sqrt[n]{n+h}\geq \sqrt[n+1]{n+1+h} \iff (n+h)^{n+1}\geq (n+1+h)^n \iff$$

$$n+h\geq\left(\frac{n+1+h}{n+h}\right)^n$$

where latter inequality follows from

$$n+h>3>e>\left(1+\frac{1}{n}\right)^n>\left(1+\frac{1}{n+h}\right)^n=\left(\frac{n+1+h}{n+h}\right)^n.$$

Remark. *The Lemma can be proved by Math Induction without reference to e.*
Note that

$$\sqrt[n]{n+h}>\sqrt[n+1]{n+1+h}\iff a_n>b_n,$$

where
$a_n:=(n+h)^{n+1}$,
$b_n:=(n+1+h)^n$.

1. Base of Math Induction.

$$a_1-b_1=(3+h)^4-(4+h)^3=(3+h)^4-(3+h)^3-3(3+h)^2-3(3+h)-1=$$

$$=(3+h)^3((3+h)-1)-3(3+h)^2-3(3+h)-1=$$

$$=(3+h)^2((3+h)(2+h)-3)-3(3+h)-1=$$

$$=(3+h)^2(h^2+5h+3)-3(3+h)-1>3(3+h)^2-3(3+h)-1=$$

$$=3(3+h)(2+h)-1>18-1=17.$$

2. Auxiliary inequality. For any $n \in \mathbb{N}$ holds inequality

$$\frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n} \iff a_{n+1}b_n > a_n b_{n+1}.$$

Indeed,

$$a_{n+1}b_n > a_n b_{n+1} \iff (n+1+h)^{n+2} \cdot (n+1+h)^n >$$

$$> (n+h)^{n+1} \cdot (n+2+h)^{n+1} \iff .$$

$$\left((n+1+h)^2\right)^{n+1} > \left((n+h)^2 + 2(n+h)\right)^{n+1} \iff$$

$$(n+h+1)^2 > (n+h)^2 + 2(n+h) \iff 1 > 0.$$

3. Step of Math Induction. For any natural $n \geq 3$ assuming $a_n > b_n$ and using inequality $\frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n}$ we obtain

$$a_{n+1} = a_n \cdot \frac{a_{n+1}}{a_n} > b_n \cdot \frac{b_{n+1}}{b_n} = b_{n+1}. \qquad \blacksquare$$

Corollary. For any $n \geq 3$ and real $h > 0$ holds inequality

$$\sqrt[3]{3+h} \geq \sqrt[n]{n+h}.$$

Now we will prove that for any $0 \leq k \leq n-3$ holds inequality

$$r_n(k) \leq \sqrt[3]{3 + \sqrt[3]{3 + \sqrt[3]{3 + \dots + \sqrt[3]{3}}}}$$

($k+1$ roots), using Math. Induction by k.

1.If $k = 0$ then

$$\sqrt[n]{n} \leq \sqrt[3]{3}.$$

2.For any k such that $1 \leq k \leq n-3$ holds

$$r_{k-1}(n) \leq \sqrt[3]{3 + \sqrt[3]{3 + \sqrt[3]{3 + \dots + \sqrt[3]{3}}}}$$

(k roots) then, applying Corollary to $h = r_{k-1}(n)$, we obtain

$$r_k(n) = \sqrt[n-k]{n-k+r_{k-1}(n)} < \sqrt[3]{3+\sqrt[3]{3+\sqrt[3]{3+\sqrt[3]{3+...+\sqrt[3]{3}}}}}(k+1 \text{ roots})$$

Let $a_1 = \sqrt[3]{3}$ and $a_{n+1} = \sqrt[3]{3+a_n}$, $n \in \mathbb{N}$ then $a_n < 2$ for any $n \in \mathbb{N}$.
Indeed, $\sqrt[3]{3} < 2$ and from supposition $a_n < 2$ we obtain

$$a_{n+1} = \sqrt[3]{3+a_n} < \sqrt[3]{3+2} = \sqrt[3]{5} < 2.$$

Hence, $r_k(n) < 2$ for any $0 \leq k \leq n-3$ and, therefore,

$$r_{n-2}(n) = \sqrt{2+r_{n-3}(n)} < \sqrt{2+2} = 2.$$

For establishing upper bounds of nested radicals represented in the next problem will be useful

Lemma 3. For any positive real a, b and any natural p, n and $k \in \{0,1,2,...,n\}$ let

$$R_k(n) := \sqrt[p]{a \cdot b^{p^{n-k}} + \sqrt[p]{a \cdot b^{p^{n-k+1}} + ... + \sqrt[p]{a \cdot b^{p^n}}}},$$

($k+1$ radicals)
Then

$$R_k(n) = b^{p^{n-k-1}} \sqrt[p]{a + \sqrt[p]{a + ... + \sqrt[p]{a}}}$$

($k+1$ radicals).

Proof. (Math Induction by $k \in \{0,1,2,...,n\}$).
First of all note that $R_k(n)$ can be defined recursively as follows:

$$R_0(n) := \sqrt[p]{a \cdot b^{p^n}},\ R_k(n) = \sqrt[p]{a \cdot b^{p^{n-k}} + R_{k-1}(n)}, k \in \{1,2,...,n\}$$

Base of M.I.

$$R_0(n) = \sqrt[p]{a \cdot b^{p^n}} = b^{p^{n-1}} \sqrt[p]{a}, R_1(n) = \sqrt[p]{a \cdot b^{p^{n-1}} + R_0(n)} = \sqrt[p]{a \cdot b^{p^{n-1}} + \sqrt[p]{a \cdot b^{p^n}}} =$$

$$= \sqrt[p]{a \cdot b^{p^{n-1}} + b^{p^{n-1}} \sqrt[p]{a}} = b^{p^{n-2}} \sqrt[p]{a + \sqrt[p]{a}}.$$

Step of Math Induction. For any $k \in \{1,2,...,n\}$ assuming

$$R_{k-1}(n) = b^{p^{n-k}} \sqrt[p]{a + \sqrt[p]{a + ... + \sqrt[p]{a}}}$$

(k radicals) we obtain

$$R_k(n)=\sqrt[p]{a\cdot b^{p^{n-k}}+R_{k-1}(n)}=\sqrt[p]{a\cdot b^{p^{n-k}}+b^{p^{n-k}}\sqrt[p]{a+\sqrt[p]{a+\ldots+\sqrt[p]{a}}}}=$$

$$=b^{p^{n-k-1}}\sqrt[p]{a+\sqrt[p]{a+\ldots+\sqrt[p]{a}}}$$

($k+1$ radicals).

Corollary 1. Let $(a_n)_{\mathbb{N}}$ be sequence of non negative real numbers such that for some positive real a and b holds inequality $a_n\le a\cdot b^{2^n}, n\in\mathbb{N}$.
Then for any $n\in\mathbb{N}, k\in\{0,1,2,\ldots,n\}$ holds inequality

(1) $$\sqrt{a_{n-k}+\sqrt{a_{n-k+1}+\ldots+\sqrt{a_n}}}\le b^{2^{n-k-1}}\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}\le M\cdot b^{2^{n-k-1}}.$$

Proof. In particular for $p=2$ from Lemmas 3 and 4 follows

$$\sqrt{a_{n-k}+\sqrt{a_{n-k+1}+\ldots+\sqrt{a_n}}}\le\sqrt{a\cdot b^{2^{n-k}}+\sqrt{a\cdot b^{2^{n-k+1}}+\ldots+\sqrt{a\cdot b^{2^n}}}}=$$

$$b^{2^{n-k-1}}\sqrt{a+\sqrt{a+\ldots+\sqrt{a}}}\le M\cdot b^{2^{n-k-1}}, \text{where } M=\frac{1+\sqrt{1+4a}}{2}$$

(see solution to Problem 2a).

Corollary 2. (Criteria of convergence $\sqrt{a_1+\sqrt{a_2+\ldots+\sqrt{a_n}}}$). Let $(a_n)_{\mathbb{N}}$ be sequence of non negative real numbers and let

$$r_n:=\sqrt{a_1+\sqrt{a_2+\ldots+\sqrt{a_n}}}.$$

Then sequence $(r_n)_{\mathbb{N}}$ is bounded from above iff $a_n\le a\cdot b^{2^n}, n\in\mathbb{N}$ for some positive a,b.

Proof. Let M be some upper bound for $(r_n)_{\mathbb{N}}$,that is $r_n\le M$ for any $n\in\mathbb{N}$ and we obtain

$$a_n^{1/2^n}=\sqrt{0+\sqrt{0+\ldots+\sqrt{0+\sqrt{a_n}}}}\le\sqrt{a_1+\sqrt{a_2+\ldots+\sqrt{a_n}}}\le M.$$

Hence, $a_n\le a\cdot b^{2^n}$, where $a=1$ and $b=M$.

If $a_n \leq a \cdot b^{2^n}, n \in \mathbb{N}$ for some positive a, b then by corollary 1 for $k = n - 1$ we obtain

$$r_n \leq M \cdot b^{2^0} = M \cdot b.$$

Problem 4. For any $n \in \mathbb{N}$ find upper bound for $n-$nested radical (contain n radicals):

a.

$$\sqrt{2^{2^0} + \sqrt{2^{2^1} + \sqrt{2^{2^2} + ... + \sqrt{2^{2^{n-1}}}}}};$$

b.

$$\sqrt{1^0 + \sqrt{2^1 + \sqrt{2^2 + ... + \sqrt{2^n}}}};$$

c.

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + ... + \sqrt{n}}}};$$

d.

$$\sqrt{1 + \sqrt{3 + \sqrt{5 + ... + \sqrt{2n - 1}}}};$$

e.

$$\sqrt{1^2 + \sqrt{2^2 + \sqrt{3^2 + ... + \sqrt{n^2}}}}$$

f.

$$\sqrt{1! + \sqrt{2! + \sqrt{3! + ... + \sqrt{n!}}}}$$

Solution.
a.Since

$$a_n = 2^{2^{n-1}} = 1 \cdot \left(\sqrt{2}\right)^{2^n}$$

then by corollary for $k = n, a = 1, b = \sqrt{2}$ we obtain

$$\sqrt{2^{2^0} + \sqrt{2^{2^1} + \sqrt{2^{2^2} + ... + \sqrt{2^{2^{n-1}}}}}} = \sqrt{2} \cdot \sqrt{1 + 1\sqrt{1 + \sqrt{1 + ... + \sqrt{1}}}} \leq$$

$$\leq \sqrt{2} \max\left\{1, \frac{1 + \sqrt{5}}{2}\right\} = \frac{\sqrt{2} + \sqrt{10}}{2} < \frac{5}{2}.$$

b. Since $n \leq 2^n, n \in \mathbb{N} \cup \{0\}$ then $2^n \leq 2^{2^{n-1}}$ and, therefore,

$$\sqrt{1^0+\sqrt{2^1+\sqrt{2^2+...+\sqrt{2^n}}}} \le \sqrt{1^0+\sqrt{2^{2^{1-1}}+\sqrt{2^{2^{2-1}}+...+\sqrt{2^{2^{n-1}}}}}} <$$

$$< \sqrt{1+\frac{\sqrt{2}+\sqrt{10}}{2}} < \sqrt{1+\frac{5}{2}} < 2.$$

c.,d. Noting that $n \le 2n-1 < 2^{2^{n-1}}$ for any $n \in \mathbb{N}$ we obtain

$$\sqrt{1+\sqrt{2+\sqrt{3+...+\sqrt{n}}}} < \sqrt{1+\sqrt{3+\sqrt{5+...+\sqrt{2n-1}}}} <$$

$$< \quad \sqrt{1^0+\sqrt{2^1+\sqrt{2^2+...+\sqrt{2^{n-1}}}}} < 2.$$

$$\sqrt{1^2+\sqrt{2^2+\sqrt{3^2+...+\sqrt{n^2}}}} < \sqrt{1+\sqrt{2^2+\sqrt{3^2+...+\sqrt{n^2}}}} <$$

$$< \quad \sqrt{2^{2^0}+\sqrt{2^{2^1}+\sqrt{2^{2^2}+...+\sqrt{2^{2^{n-1}}}}}}$$

e. Since $2n-1 \le n^2 < 2^{2^{n-1}}$ for any $n \in \mathbb{N}$ (because $n^2 \le 2^n$ for $n \ge 4$, implies $2^n \le 2^{2^{n-1}}$ for any $n \in \mathbb{N} \cup \{0\}$ and obviously $n^2 \le 2^{2^{n-1}}$ for $n = 1, 2, 3$) then

$$\sqrt{1^2+\sqrt{2^2+\sqrt{3^2+...+\sqrt{n^2}}}} < \sqrt{2^{2^0}+\sqrt{2^{2^1}+\sqrt{2^{2^2}+...+\sqrt{2^{2^{n-1}}}}}} < \frac{5}{2}.$$

f.. First note that for any $n \in \mathbb{N}$ holds inequality $n! < 2^{2^{n-1}}$. Indeed,

$$\frac{(n+1)!}{n!} \le \frac{2^{2^n}}{2^{2^{n-1}}} \iff n+1 \le 2^{2^{n-1}}$$

for any $n \in \mathbb{N}$. Then since $1! < 2^{2^{1-1}} = 2$ and $n! < 2^{2^{n-1}}$ implies

$$(n+1)! = n! \cdot \frac{(n+1)!}{n!} < 2^{2^{n-1}} \cdot \frac{2^{2^n}}{2^{2^{n-1}}} = 2^{2^n}$$

we conclude by Math Induction that $n! < 2^{2^{n-1}}, n \in \mathbb{N}$.
Hence,

$$\sqrt{1!+\sqrt{2!+\sqrt{3!+...+\sqrt{n!}}}} < \sqrt{1+\sqrt{2^{2^{1-1}}+\sqrt{2^{2^{2-1}}+...+\sqrt{2^{2^{n-1}}}}}} < 2.$$

Problem 5. Let

$$a_n := \sqrt[3]{1+\sqrt[3]{2+\sqrt[3]{3+\sqrt[3]{4+...+\sqrt[3]{n}}}}}, n \in \mathbb{N}.$$

Prove that:
(1) $a_{n+1}^3 < 1+\sqrt[3]{2}\cdot a_n$for any$n \in \mathbb{N}$
(2) Sequence $(a_n)_{\mathbb{N}}$ is convergent.
Solution.
1. Noting that $k \le 2^{3^{k-2}}(k-1)$ for any $k \in \mathbb{N}\setminus\{1\}$ (equality holds only if $k=2$) we obtain

$$a_{n+1}^3 = 1+\sqrt[3]{2+\sqrt[3]{3+\sqrt[3]{4+...+\sqrt[3]{n+\sqrt[3]{n+1}}}}} <$$

$$1+\sqrt[3]{2+\sqrt[3]{2^{3^{3-2}}\cdot 2+\sqrt[3]{2^{3^{4-2}}\cdot 3+...+\sqrt[3]{2^{3^{n-2}}(n-1)+\sqrt[3]{2^{3^{n-1}}\cdot n}}}}} =$$

$$1+\sqrt[3]{2+\sqrt[3]{2^{3^{3-2}}\cdot 2+\sqrt[3]{2^{3^{4-2}}\cdot 3+...+\sqrt[3]{2^{3^{n-2}}(n-1)+2^{3^{n-2}}\sqrt[3]{n}}}}} =$$

$$1+\sqrt[3]{2+\sqrt[3]{2^{3^{3-2}}\cdot 2+\sqrt[3]{2^{3^{4-2}}\cdot 3+...+2^{3^{n-3}}\sqrt[3]{(n-1)+\sqrt[3]{n}}}}} = ...$$

$$= 1+\sqrt[3]{2+2\sqrt[3]{2+\sqrt[3]{3+...+\sqrt[3]{(n-1)+\sqrt[3]{n}}}}} = 1+\sqrt[3]{2}\cdot a_n.$$

2. First we will prove that $a_n < \sqrt[3]{4}$for any $n \in \mathbb{N}$.
Indeed, $a_1 = 1 < \sqrt[3]{4}$ and

$$a_2 = \sqrt[3]{1+\sqrt[3]{2}} < \sqrt[3]{4} \iff \sqrt[3]{2} < 3.$$

For any $n \in \mathbb{N}$ assuming $a_n < \sqrt[3]{4}$ we obtain

$$a_{n+1}^3 < 1+\sqrt[3]{2}\cdot a_n < 1+\sqrt[3]{2}\cdot\sqrt[3]{4} = 3$$

and, therefore,

$$a_{n+1} < \sqrt[3]{3} < \sqrt[3]{4}.$$

Thus, by Math Induction, $a_n < \sqrt[3]{4}$for any $n \in \mathbb{N}$ and since $a_{n+1} > a_n$ for any $n \in \mathbb{N}$ we can conclude that sequence $(a_n)_{\mathbb{N}}$ is convergent as increasing and bounded from above.

Another solution of 1. Note that $n \leq 2^{3^{n-2}}(n-1)$ for any $n \in \mathbb{N} \setminus \{1\}$ (equality holds only if $n = 2$)
For any $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ such that $k \leq n$ let $r_0(n) := 0$,and

$$r_{k+1}(n) = \sqrt[3]{n - k + r_k(n)}, k \in \{0, 1, 2, ..., n\}.$$

Then $a_n = r_n(n), \forall n \in \mathbb{N}$.
Also note that

$$a_{n+1} = r_{n+1}(n+1) = \sqrt[3]{1 + r_n(n+1)}.$$

Note that

$$r_1(n+1) = \sqrt[3]{n+1} < \sqrt[3]{2^{3^{n+1-2}} \cdot (n+1-1)} =$$

$$= 2^{3^{n-2}} \sqrt[3]{n} = 2^{3^{n-2}} r_1(n).$$

Let $1 \leq k \leq n$ be any. Assuming $r_k(n+1) < 2^{3^{n-k-1}} r_k(n)$ and since $n + 1 - k \leq 2^{3^{n-k-1}}(n-k)$ for any $k = 0, 1, ..., n-1$(equality holds only if $k = n - 1$) we obtain

$$r_{k+1}(n+1) = \sqrt[3]{n+1-k+r_k(n+1)} < \sqrt[3]{2^{3^{n-k-1}}(n-k) + 2^{3^{n-k-1}} r_k(n)} =$$

$$= 2^{3^{n-k-2}} \sqrt[3]{(n-k) + r_k(n)} = 2^{3^{n-(k+1)-1}} r_{k+1}(n).$$

Thus, by Math Induction we proved $r_k(n+1) < 2^{3^{n-1-k}} r_k(n)$ for any $0 \leq k \leq n$. In particular for $k = n$ we have

$$a_{n+1} = r_{n+1}(n+1) = \sqrt[3]{1 + r_n(n+1)} <$$

$$< \sqrt[3]{1 + 2^{3^{n-1-n}} r_n(n)} = \sqrt[3]{1 + 2^{3^{-1}} r_n(n)} = \sqrt[3]{1 + \sqrt[3]{2} a_n}.$$

Thus, $a_{n+1}^3 < 1 + \sqrt[3]{2} \cdot a_n$ for any $n \in \mathbb{N}$.

2. Infinite nested square roots.

As usually we start from concrete problems which motivate consideration of situation represented in this problems in general.

* ★ **Problem 1.** Let

$$r_n := \sqrt{1 + f_1\sqrt{1 + f_2\sqrt{1 + f_3\sqrt{1 + ...f_n\sqrt{1}}}}},$$

where f_n be $n - th$ Fibonacci number defined by $f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N}$ and $f_0 = 0,\ f_1 = 1$.
Prove that sequence (r_n) is convergent and find $r := \lim\limits_{n\to\infty} r_n$, that is find the value of infinite nested root

$$r = \sqrt{1 + f_1\sqrt{1 + f_2\sqrt{1 + f_3\sqrt{1 + ... + .f_n\sqrt{...}}}}}$$

Solution. First we will find the sum

$$f_1 \cdot q + f_2 \cdot q^2 + ... + f_n \cdot q^n.$$

Let

$$S_n(q) := \sum_{k=1}^{n} f_k q^k$$

and

$$S(q) = \sum_{n=1}^{\infty} f_n q^n$$

Since

$$\Delta\left(f_k \cdot q^k\right) = f_{k+1}q^{k+1} - f_k q^k = f_k q^{k+1} + f_{k-1}q^{k+1} - f_k q^k =$$

$$= (q-1)\, q^k f_k + q^{k-1} f_{k-1} \cdot q^2$$

then

$$f_{n+1} \cdot q^{n+1} - f_1 q = \sum_{k=1}^{n}\left(f_{k+1}q^{k+1} - f_k q^k\right) = (q-1)\sum_{k=1}^{n} q^k f_k + q^2 \sum_{k=1}^{n} q^{k-1} f_{k-1} =$$

$$= (q-1)\, S_n(q) + q^2 \sum_{k=1}^{n-1} q^k f_k = (q-1)\, S_n(q) + q^2\left(\sum_{k=1}^{n} q^k f_k - q^n f_n\right) =$$

$$= (q-1)\, S_n(q) + q^2\left(S_n(q) - q^n f_n\right) = \left(q^2 + q - 1\right) S_n(q) - q^{n+2} f_n.$$

Hence,

$$\left(q^2 + q - 1\right) S_n(q) = f_{n+1} \cdot q^{n+1} - f_1 q + q^{n+2} f_n \iff$$

$$S_n(q) = \frac{f_{n+1} \cdot q^{n+1} - f_1 q + q^{n+2} f_n}{q^2 + q - 1} \iff S_n(q) = \frac{f_1 q - q^{n+2} f_n - q^{n+1} f_{n+1}}{1 - q - q^2}.$$

Since $\lim\limits_{n\to\infty} \sqrt[n]{f_n} = \phi$ then radius of convergency $S(q)$ equal $\dfrac{1}{\phi} = \dfrac{\sqrt{5}-1}{2} = -\overline{\phi}$.
If $|q| < \dfrac{\sqrt{5}-1}{2}$ then

$$\lim_{n\to\infty} q^{n+2} f_n = \lim_{n\to\infty} q^{n+1} f_{n+1} = 0$$

and, therefore,

$$S(q) = \sum_{n=1}^{\infty} f_n q^n = \frac{q}{1-q-q^2}$$

for any such q.
In particular

$$S_n\left(\frac{1}{2}\right) = \frac{f_1}{2} + \frac{f_2}{2^2} + ... + \frac{f_n}{2^n} = \frac{1/2 - f_n/2^{n+2} - f_{n+1}/2^{n+1}}{1 - 1/2 - (1/2)^2} =$$
$$= 2 - \frac{f_n}{2^n} - \frac{f_{n+1}}{2^{n-1}} < 2.$$

Note that

$$r_n = \sqrt{1 + f_1\sqrt{1 + f_2\sqrt{1 + f_3\sqrt{1 + ... + .f_n\sqrt{1}}}}} =$$
$$= \sqrt{1 + \sqrt{c_1 + \sqrt{c_2 + \sqrt{c_3 + ... + .\sqrt{c_n}}}}}$$

where $c_n = f_1^{2^n} f_2^{2^{n-1}} ... f_n^{2^1}$ $(c_0 = 1)$
Since by weighted AM-GM Inequality

$$\sqrt[2^{n+1}]{c_n} = f_1^{1/2} f_2^{1/2^2} ... f_n^{1/2^n} < \frac{1}{2}\cdot f_1 + \frac{1}{2^2}\cdot f_2 + ... + \frac{1}{2^n}\cdot f_n = S_n\left(\frac{1}{2}\right)$$
$$\text{and } S_n\left(\frac{1}{2}\right) < 2$$

then $c_n < 2^{2^{n+1}}$ and, therefore, sequence $(r_n)_{\mathbb{N}}$ is convergent by

Corollary 2.(Criteria of convergency of $x_n = \sqrt{a_1 + \sqrt{a_2 + ... + \sqrt{a_n}}}$).
Numerical experiments give us $r_1 = 1.4142, r_2 = 1.5538, r_3 = 1.6288, ...,$
$r_{15} = 1.7531, r_{16} = 1.755, r_{17} = 1.7551$
So, infinite nested root

$$\sqrt{1 + f_1\sqrt{1 + f_2\sqrt{1 + f_3\sqrt{1 + ... + .f_n\sqrt{1 + ...}}}}}$$

define numerical constant which approximately equal 1.755.
Remains the question: Can be this constant expressed via already well known constants?

Problem 2 (Problem.(2062.Proposed by K.R.S. Sastry, Dodballapur, India). Find a positive integer n so that both the continued roots

$$\sqrt{1995+\sqrt{n+\sqrt{1995+\sqrt{n+...}}}}$$

and

$$\sqrt{n+\sqrt{1995+\sqrt{n+\sqrt{1995+...}}}}$$

converge to positive integers.
We will return to solving this problem later, having first studied the behavior of the sequence

$$x_n := \sqrt{a+\sqrt{b+\sqrt{a+...+\sqrt{\cdot\frac{a+b+(-1)^{n+1}(a-b)}{2}}}}}$$

(n roots), $n \in \mathbb{N}$ where a and b be positive real numbers.
The sequence $(x_n)_{\mathbb{N}}$ can be defined recursively as follows:

$$x_1 = \sqrt{a}, x_2 = \sqrt{a+\sqrt{b}},\ x_{n+2} = \sqrt{a+\sqrt{b+x_n}}, n \in \mathbb{N}.$$

Let $h(x) := \sqrt{a+\sqrt{b+x}}$. Then

$$x_{n+2} = h(x_n), n \in \mathbb{N}.$$

Since $x_1 < x_2 < x_3$ and for any $n \in \mathbb{N}$, assuming $x_{2n-1} < x_{2n} < x_{2n+1}$ we obtain

$$h(x_{2n-1}) < h(x_{2n}) < h(x_{2n+1}) \iff x_{2n+1} < x_{2n+2} < x_{2n+3}.$$

Thus, by Math Induction proved that $x_n < x_{n+1}$ for any $n \in \mathbb{N}$.

Let $m := \max\{a, b\}$ and $m_n = \sqrt{m+\sqrt{m+\sqrt{m...+\sqrt{m}}}}$(n roots).

Since $x_n \le m_n, \forall n \in \mathbb{N}$ and $m_n \le \dfrac{1+\sqrt{4m+1}}{2}$ then (x_n) is bounded from above and, therefore, $(x_n)_{\mathbb{N}}$ is convergent as increasing sequence.
Let $x := \lim\limits_{n\to\infty} x_n > \sqrt{a}$. Then

$$x = \lim_{n\to\infty} h(x_n) = h\left(\lim_{n\to\infty} x_n\right) = h(x) \Leftrightarrow$$

$$\sqrt{a+\sqrt{b+x}}=x \iff \left(x^2-a\right)^2=x+b \iff \left(x-\frac{a}{x}\right)^2=\frac{1}{x}+\frac{b}{x^2}$$

Note that $\left(x-\dfrac{a}{x}\right)^2$ strictly increase in $(\sqrt{a},\infty)$ (because $x-\dfrac{a}{x}>0$ for $x>\sqrt{a}$ and increase in $(0,\infty)$) and $\dfrac{1}{x}+\dfrac{b}{x^2}$ strictly decrease.
Hence, since $\left(x-\dfrac{a}{x}\right)^2-\left(\dfrac{1}{x}+\dfrac{b}{x^2}\right)$ is negative for $x=\sqrt{a}$ and it is positive for big enough positive x then equation $\left(x-\dfrac{a}{x}\right)^2=\dfrac{1}{x}+\dfrac{b}{x^2}$ has a unique solution on $(\sqrt{a},\infty)$.
So, infinite nested root

$$\sqrt{a+\sqrt{b+\sqrt{a+\sqrt{b+\ldots.}}}}\quad \lim_{n\to\infty} x_n=x,$$

where x is unique solution of equation $x^4-2x^2a-x+a^2-b=0$ in $(\sqrt{a},\infty)$.
Together with infinite nested root

$$\sqrt{a+\sqrt{b+\sqrt{a+\sqrt{b+\ldots.}}}}$$

we also will consider nested root

$$\sqrt{b+\sqrt{a+\sqrt{b+\sqrt{a+\ldots.}}}}$$

which is defined as limit of sequence (y_n) defined recursively by

$$y_1=\sqrt{b}, y_2=\sqrt{b+\sqrt{a}}, y_{n+2}=\sqrt{b+\sqrt{a+y_n}}, n\in\mathbb{N}.$$

But, some times more convenient simultaneous definition sequences $(x_n),(y_n)$ by the following system of recurrences
(R)

$$\begin{cases} x_{n+1}=\sqrt{a+y_n} \\ y_{n+1}=\sqrt{b+x_n} \end{cases}, n\in\mathbb{N}$$

with initial conditions $x_1=\sqrt{a}$ and $y_1=\sqrt{b}$. As follows from the proved above both sequences are convergent and,
therefore, $x:=\lim\limits_{n\to\infty} x_n>\sqrt{a}, y:=\lim\limits_{n\to\infty} y_n>\sqrt{b}$ satisfies to system of equations
(E)

$$\begin{cases} x=\sqrt{a+y} \\ y=\sqrt{b+x}. \end{cases}$$

Now we came back to solution of the Problem 1.

Solution. Consider two sequences $(x_n), (y_n)$ defined by the system of recurrences (R) for $a = 1995$ and $b = n$.
Then

$$x = \sqrt{1995 + \sqrt{n + \sqrt{1995 + \sqrt{n + ...}}}}$$

and

$$y = \sqrt{n + \sqrt{1995 + \sqrt{n + \sqrt{1995 + ...}}}}$$

are solution of the system

$$\begin{cases} x = \sqrt{1995 + y} \\ y = \sqrt{n + x}. \end{cases} \iff \begin{cases} x^2 = 1995 + y \\ y^2 = n + x \end{cases}.$$

Let $y \in \mathbb{N}$ be such that $1995 + y$ is a perfect square, that is $1995 + y = (44 + t)^2$.
Then

$$x = 44 + t,\ y = x^2 - 1995 = (44 + t)^2 - 1995 = t^2 + 88t - 59$$

and

$$n = y^2 - x = \left(t^2 + 88t - 59\right)^2 - (44 + t) =$$

$$= \quad t^4 + 176t^3 + 7626t^2 - 10\,385t + 3437$$

for any $t \in \mathbb{N}$
(because $P(t) := t^4 + 176t^3 + 7626t^2 - 10\,385t + 3437 \geq 1$ for any $t \in \mathbb{N}$).
Thus, for any $t \in \mathbb{N}$ we have

$$(x, y, n) = \left(44 + t, t^2 + 88t - 59, P(t)\right)$$

For example for $t = 1$ we obtain $x = 45, y = 84, n = P(t) = 855$.

Remark. More general nested root

$$z_n := \sqrt{p + r\sqrt{q + r\sqrt{p + ... + r\sqrt{.\frac{p + q + (-1)^{n+1}(p - q)}{2}}}}}, n \in \mathbb{N},\ p, q, r > 0$$

can be reduced to nested root x_n,considered above.
Indeed, since

$$\frac{z_n}{r^2} = \sqrt{\frac{p}{r^2} + \sqrt{\frac{q}{r^2} + \sqrt{\frac{p}{r^2} + ... + \sqrt{.\frac{p/r^2 + q/r^2 + (-1)^{n+1}(p/r^2 - q/r^2)}{2}}}}}$$

then denoting $x_n := \frac{z_n}{r^2}, a := \frac{p}{r^2}, b := \frac{q}{r^2}$ we obtain

$$x_n := \sqrt{a + \sqrt{b + \sqrt{a + ... + \sqrt{\cdot \frac{a + b + (-1)^{n+1}(a-b)}{2}}}}}, n \in \mathbb{N}.$$

Problem 3. Explore convergence and find limit of sequence (a_n):

a) $a_{n+2} = \sqrt{7 - \sqrt{7 + a_n}}, n \in \mathbb{N}$ and $a_1 = \sqrt{7}, a_2 = \sqrt{7 - \sqrt{7}}$;

b) $a_{n+2} = \sqrt{19 - \sqrt{5 + a_n}}, n \in \mathbb{N}$ and $a_1 = \sqrt{19}, a_2 = \sqrt{19 - \sqrt{5}}$;

c) $a_{n+2} = \sqrt{9 - \sqrt{23 + a_n}}, n \in \mathbb{N}$ and $a_1 = \sqrt{9}, a_2 = \sqrt{9 - \sqrt{23}}$.

And again, instead solving all these problems we will explore situation in general, namely for given positive real numbers a, b such that $a^2 > b$ we will consider two sequences (x_n) and (y_n) defined recursively

$x_{n+2} = \sqrt{a - \sqrt{b + x_n}},\ n \in \mathbb{N},$ where $x_1 = \sqrt{a}, x_2 = \sqrt{a - \sqrt{b}}$

and

$y_{n+2} = \sqrt{b + \sqrt{a - y_n}},\ n \in \mathbb{N},$ where $y_1 = \sqrt{b},\ y_2 = \sqrt{b + \sqrt{a}}$.

Both sequences can be defined by the following system of recurrences of the first order:

(S)

$$\begin{cases} x_{n+1} = \sqrt{a - y_n} \\ y_{n+1} = \sqrt{b + x_n} \end{cases}, n \in \mathbb{N}$$

and $x_1 = \sqrt{a}, y_1 = \sqrt{b}$.

Let $\alpha(t) := \sqrt{a - t}, \beta(t) := \sqrt{b + t}$ and $\varphi(t) := \alpha(\beta(t)) = \sqrt{a - \sqrt{b + t}}$, $\psi(t) := \beta(\alpha(t)) = \sqrt{b + \sqrt{a - t}}$.

Then

(S')

$$\begin{cases} x_{n+1} = \alpha(y_n) \\ y_{n+1} = \beta(x_n) \end{cases}, n \in \mathbb{N} \cup \{0\}$$

and $x_0 = y_0 = 0$.

and $x_{n+2} = \varphi(x_n), y_{n+2} = \psi(y_n), n \in \mathbb{N}$ where $x_1 = \sqrt{a}, x_2 = \sqrt{a - \sqrt{b}}$

and $y_{n+2} = \psi(y_n), n \in \mathbb{N}$, where $y_1 = \sqrt{b},\ y_2 = \sqrt{b + \sqrt{a}}$.

Since $\varphi(t)$ is defined and decrease on $I := (0, a^2 - b)$ then for $t \in I$

$$0 = \varphi(a^2 - b) < \varphi(t) < \varphi(0) = \sqrt{a - \sqrt{b}},$$

that is $\varphi(I) = \left(0, \sqrt{a - \sqrt{b}}\right)$.

To provide existence of x_n for any $n \in \mathbb{N}$ we should claim

$$\varphi(I) \subset I \iff \sqrt{a - \sqrt{b}} < a^2 - b \iff 1 < (a^2 - b)\left(a + \sqrt{b}\right)$$

and

$$x_1 \in I \iff \sqrt{a} < a^2 - b \iff b < a^2 - \sqrt{a}.$$

Thus, for further we assume that positive a, b satisfies to inequalities

(1) $1 < \left(a^2 - b\right)\left(a + \sqrt{b}\right)$ and

(2) $b < a^2 - \sqrt{a}$.

Assuming that both sequences are convergent and denoting
$x := \lim_{n\to\infty} x_n, y := \lim_{n\to\infty} y_n$ we will consider system of equations

$$\begin{cases} x = \alpha(y) \\ y = \beta(x) \end{cases} \iff \begin{cases} x = \varphi(x) \\ y = \psi(y) \end{cases}.$$

Let

$$h(t) := t - \varphi(t) = t - \sqrt{a - \sqrt{b + t}}.$$

Note that $h(t)$ is increasing function on $(0, \sqrt{a})$ and also note that $\alpha(t), \psi(t)$ are decreasing functions on $(0, \sqrt{a})$ and $\beta(t)$ is increasing function.
Since

$$h(0) = -\varphi(0) = -\sqrt{a - \sqrt{b}} < 0$$

and

$$h\left(\sqrt{a}\right) = h(x_1) = x_1 - \varphi(x_1) =$$

$$= x_1 - x_3 > 0$$

(because $x_1 > x_n$ for any $n > 1$ and in particular $x_1 > x_3$)
then there is solution of equation $x = \varphi(x)$ on $(0, \sqrt{a})$ and this solution is unique because $h(x) := x - \varphi(x)$ is increasing function on
$(0, \sqrt{a}) = (x_0, x_1)$.
Denoting this solution via x_* and denoting $y_* := \beta(x_*)$ we obtain two identities
$x_* = \varphi(x_*), y_* = \psi(y_*)$.
Note that $x_0 < x_* < x_1$ implies

$$\beta(x_0) < \beta(x_*) < \beta(x_1) \iff y_1 < y_* < y_2$$

and

$$\varphi(x_0) < \varphi(x_*) < \varphi(x_1) \iff x_3 < x_* < x_2.$$

Before moving further and taking in account that $x_1 > x_2 > x_3$ we will prove (using Math Induction) that inequality $x_n > x_3$ also holds for any $n \geq 4$.
We have

$$x_1 > x_2 > x_3 \implies \varphi(x_1) < \varphi(x_2) < \varphi(x_3) \iff$$

$$x_3 < x_4 < x_5$$

and noting that $\varphi_2(t) := \varphi(\varphi(t))$ increase on I we obtain

$$x_0 < x_2 \implies \varphi_2(x_0) < \varphi_2(x_2) \iff x_4 < x_6$$

and

$$x_0 < x_3 \implies \varphi_2(x_0) < \varphi_2(x_3) \iff x_4 < x_7.$$

Hence, $x_4, x_5, x_6, x_7 > x_3$ and for any $n \geq 4$ assuming $x_n, x_{n+1}, x_{n+2}, x_{n+3} > x_3$ we obtain

$$x_{k+4} = \varphi_2(x_k) > \varphi_2(x_3) = x_7 > x_3, k = n, n+1, n+2, n+3.$$

Thus, $x_n \geq x_3$ for any $n \in \mathbb{N}$ with equality only if $n = 3$.
Also note that for any $n \in \mathbb{N}$ obviously holds inequality
$y_n = \sqrt{b + x_{n-1}} \geq \sqrt{b} = y_1$ with equality only if $n = 1$.
Since $x_3 < x_*$ and for any $n \in \mathbb{N}$ holds inequalities $x_3 \leq x_n$ and , $y_1 \leq y_n$ and $y_1 < x$ then

$$|x_{n+2} - x_*| = \frac{|x_{n+2}^2 - x_*^2|}{x_{n+2} + x_*} = \frac{|y_{n+1} - y_*|}{x_{n+2} + x_*} =$$

$$= \frac{|x_n - x_*|}{(x_{n+2} + x_*)(y_{n+1} + y_*)} < \frac{|x_n - x_*|}{4x_3 y_1} = \frac{|x_n - x_*|}{4\left(\sqrt{a - \sqrt{b+a}}\right)\sqrt{b}}.$$

If

$$4\left(\sqrt{a - \sqrt{b+a}}\right)\sqrt{b} > 1$$

then from

$$|x_{n+2} - x_*| < \frac{|x_n - x_*|}{4\left(\sqrt{a - \sqrt{b+a}}\right)\sqrt{b}}$$

immediately follows that (x_n) is convergent sequence.
Thus, if $4\left(\sqrt{a - \sqrt{b+a}}\right)\sqrt{b} > 1$ then $\lim_{n\to\infty} x_n = x_*$ and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \sqrt{b + x_{n-1}} = \sqrt{b + x_*} = y_*.$$

Thus, proved the

Theorem. If two positive real numbers a, b satisfies to inequalities **(1)**, **(2)** and **(3)**

$$\left(a - \sqrt{b+a}\right) b > 1/16$$

then sequences (x_n) and (y_n) defined recursively by system of recurrences (S) both convergent and positive solution (x_*, y_*) of the system $\begin{cases} x = \sqrt{a - y} \\ y = \sqrt{b + x} \end{cases}$ are their limits, respectively.

Consider application of the Theorem to Problem 3.
a) For $a = b = 7$ we have

$$\left(a^2 - b\right)\left(a + \sqrt{b}\right) - 1 = \left(7^2 - 7\right)\left(7 + \sqrt{7}\right) - 1 =$$

$$= 42\,(7 + 2) - 1 = 377,\; a^2 - \sqrt{a} - b = 7^2 - \sqrt{7} - 7 > 39$$

and

$$16\left(a - \sqrt{b + a}\right) b - 1 = 16\left(7 - \sqrt{14}\right) 7 - 1 > 16\,(7 - 4)\,7 - 1 = 335.$$

Also, since

$$\begin{cases} x = \sqrt{7 - y} \\ y = \sqrt{7 + x} \end{cases} \iff \begin{cases} x = 2 \\ y = 3 \end{cases}$$

then

$$\lim_{n\to\infty} x_n = 2,\; \lim_{n\to\infty} y_n = 3.$$

b) For $a = 19, b = 5$ inequalities **(1)** ,**(2)** obviously holds and

$$16\left(a - \sqrt{b + a}\right) b - 1 = 16\left(19 - \sqrt{24}\right) 7 - 1 > 16\,(19 - 5)\,7 - 1 = 1567.$$

Also, since

$$\begin{cases} x = \sqrt{19 - y} \\ y = \sqrt{5 + x} \end{cases} \iff \begin{cases} x = 4 \\ y = 3 \end{cases}$$

then

$$\lim_{n\to\infty} x_n = 4,\; \lim_{n\to\infty} y_n = 3.$$

c) For $a = 9, b = 23$ inequalities **(1)** ,**(2)** obviously holds and

$$16\left(a - \sqrt{b + a}\right) b - 1 = 16\,(9 - \sqrt{23 + 9})\,23 - 1 > 16\,(7 - 6)\,7 - 1 = 111$$

Also, since

$$\begin{cases} x = \sqrt{9 - y} \\ y = \sqrt{23 + x} \end{cases} \iff \begin{cases} x = 2 \\ y = 5 \end{cases}$$

then

$$\lim_{n\to\infty} x_n = 2,\; \lim_{n\to\infty} y_n = 5.$$

Remark. *Consider now for positive a, b, c following kind of nested roots*

$$\sqrt{a - c\sqrt{b + c\sqrt{a - c\sqrt{b + c\sqrt{a + \ldots}}}}}$$

$$\sqrt{b+c\sqrt{a-c\sqrt{b+c\sqrt{a-\gamma\sqrt{b+........}}}}}$$

or more precisely two sequences (a_n) and (b_n) which defined by system of recurrences:

(i)

$$\begin{cases} a_{n+1} = \sqrt{a - cb_n} \\ b_{n+1} = \sqrt{\beta + ca_n} \end{cases}, n \in \mathbb{N}$$

and $a_1 = \sqrt{a}, b_1 = \sqrt{b}$.

Since **(i)** $\iff \begin{cases} \dfrac{a_n}{c} = \sqrt{\dfrac{a}{c^2} - \dfrac{b_{n-1}}{c}} \\ \dfrac{b_n}{c} = \sqrt{\dfrac{b}{c^2} + \dfrac{\alpha_{n-1}}{c}} \end{cases}$ then using notations

$x_n = \dfrac{a_n}{c}, y_n = \dfrac{b_n}{c}, a = \dfrac{\alpha}{c^2}, b = \dfrac{b}{c^2}$ we can reduce exploration of sequences (a_n) and (b_n) sequences (x_n) and (y_n) defined by
(ii)

$$\begin{cases} x_{n+1} = \sqrt{a - y_n} \\ y_{n+1} = \sqrt{b + x_n} \end{cases}, n \in \mathbb{N}$$

and $x_1 = \sqrt{a}, y_1 = \sqrt{b}$. and considered above.

Problem 4.(Ramanujan's nested square roots.) Prove that

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+....}}}};$$

Problem 5. (CRUX#2222). Calculate the infinite nested root:

$$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+.....}}}}$$

And we will solve them both as one problem in the following generalized formulation:
Let $b_n = b + an, n \in \mathbb{N} \cup \{0\}$ where $a, b > 0$ and let

$$r_n := \sqrt{a^2 + b_0\sqrt{a^2 + b_1\sqrt{a^2 + b_2\sqrt{a^2 + ...b_{n-1}\sqrt{a^2}}}}}, n \in \mathbb{N}$$

Prove that sequence (r_n) converge and find $r := \lim_{n\to\infty} r_n$, i.e. find the value of infinite nested root

$$r = \sqrt{a^2 + b_0\sqrt{a^2 + b_1\sqrt{a^2 + b_2\sqrt{a^2 + ...b_{n-1}\sqrt{a^2 + ...}}}}}$$

Solution. Obvious that $r_{n+1} > r_n$ for any $n \in \mathbb{N}$ and we will prove that (r_n) have upper bound, more definitely, that $r_n < b_1$ for any $n \in \mathbb{N}$.
For any natural k and n denote

$$r_n(k) := \sqrt{a^2 + b_{k-1}\sqrt{a^2 + b_k\sqrt{a^2 + b_{k+1}\sqrt{a^2 + ...b_{k+n-2}\sqrt{a^2}}}}}.$$

Then

$$r_n(k) = \sqrt{a^2 + b_{k-1}r_{n-1}(k+1)}$$

Note, that for any $n \in \mathbb{N}$ holds identity
(1)

$$b_n^2 = a^2 + b_{n-1}b_{n+1}.$$

Indeed,

$$b_n^2 - a^2 = (b_n - a)(b_n + a) = b_{n-1}b_{n+1}.$$

Using Math. Induction by n and identity **(1)** we will prove that $r_n(k) < b_k$ for any natural n and k.

1. Base of induction. Let $n = 1$.Since $b_{k+1} > b_1 = b + a > a$ then

$$r_1(k) = \sqrt{a^2 + b_{k-1}\sqrt{a^2}} = \sqrt{a^2 + b_{k-1}a} < \sqrt{a^2 + b_{k-1}b_{k+1}} = \sqrt{b_k^2} = b_k.$$

2. Step of induction. For any $n \in \mathbb{N}$, assuming that inequality $r_n(m) < b_m$ holds for any $m \in \mathbb{N}$, we obtain

$$r_{n+1}(k) = \sqrt{a^2 + b_{k-1}r_n(k+1)} < \sqrt{a^2 + b_{k-1}b_{k+1}} = b_k.$$

Thus, in particularly we have $r_n = r_n(1) < b_1$ and, therefore,
$(r_n)_{\mathbb{N}}$ is convergent sequence.
Moreover, we will prove that $\lim_{n\to\infty} r_n(k) = b_k$ for any $k \in \mathbb{N}$.
We have

$$b_k - r_n(k) = \frac{b_k^2 - r_n^2(k)}{b_k + r_n(k)} = \frac{a^2 + b_{k-1}b_{k+1} - \left(a^2 + b_{k-1}r_{n-1}(k+1)\right)}{b_k + r_n(k)} =$$

$$= \frac{b_{k-1}\left(b_{k+1} - r_{n-1}(k+1)\right)}{b_k + r_n(k)} = \frac{b_{k-1}b_k\left(b_{k+2} - r_{n-2}(k+2)\right)}{\left(b_k + r_n(k)\right)\left(b_{k+1} + r_{n-1}(k+1)\right)} = ...$$

$$= \frac{b_{k-1}b_k...b_{k+n-3}\left(b_{k+n-1} - r_1(k+n-1)\right)}{\left(b_k + r_n(k)\right)\left(r_{n-1}(k+1) + b_{k+1}\right)...\left(b_{k+n-2} + r_2(k+n-2)\right)} =$$

$$= \frac{b_{k-1}b_k...b_{k+n-3}\left(a^2+b_{k+n-2}b_{k+n}-a^2-b_{k+n-2}a\right)}{\left(b_k+r_n\left(k\right)\right)\left(r_{n-1}\left(k+1\right)+b_{k+1}\right)...\left(b_{k+n-2}+r_2\left(k+n-2\right)\right)\left(b_{k+n-1}+r_1\left(k+n-1\right)\right)} =$$

$$= \frac{b_{k-1}b_k...b_{k+n-3}\left(b_{k+n-2}b_{k+n}-b_{k+n-2}a\right)}{\left(b_k+r_n\left(k\right)\right)\left(r_{n-1}\left(k+1\right)+b_{k+1}\right)...\left(b_{k+n-2}+r_2\left(k+n-2\right)\right)\left(b_{k+n-1}+r_1\left(k+n-1\right)\right)} =$$

$$= \frac{b_{k-1}b_k...b_{k+n-3}b_{k+n-2}b_{k+n-1}}{\left(b_k+r_n\left(k\right)\right)\left(r_{n-1}\left(k+1\right)+b_{k+1}\right)...\left(b_{k+n-2}+r_2\left(k+n-2\right)\right)\left(b_{k+n-1}+r_1\left(k+n-1\right)\right)}$$

and since $r_n\left(k\right)>a$ for any $n,k\in\mathbb{N}$ then

$$r_n\left(k\right)-b_k<\frac{b_{k-1}b_k...b_{k+n-3}b_{k+n-2}b_{k+n-1}}{\left(b_k+a\right)\left(b_{k+1}+a\right)...\left(b_{k+n-2}+a\right)\left(b_{k+n-1}+a\right)}=$$

$$=\frac{b_{k-1}b_k...b_{k+n-2}b_{k+n-1}}{b_{k+1}b_{k+2}...b_{k+n-1}b_{k+n}}=\frac{b_{k-1}b_k}{b_{k+n}}$$

Thus, $0<r_n\left(k\right)-b_k<\dfrac{b_{k-1}b_k}{b_{k+n}}$ and $\lim\limits_{n\to\infty}\dfrac{b_{k-1}b_k}{b_{k+n}}=0$
implies

$$\lim_{n\to\infty}\left(r_n\left(k\right)-b_k\right)=0$$

To be continued...

* Sign ★ before a problem means that it proposed by author of these notes.

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